Effective actions for finite temperature Lattice Gauge Theories.

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We consider a lattice gauge theory at finite temperature in (d+1) dimensions with the Wilson action and different couplings β_t and β_s for timelike and spacelike plaquettes. By using the character expansion and Schwinger-Dyson type equations we construct, order by order in β_s, an effective action for the Polyakov loops which is exact to all orders in β_t. As an example we construct the first non-trivial order in β_s for the (3+1) dimensional SU(2) model and use this effective action to extract the deconfinement temperature of the model.

1. INTRODUCTION

All the relevant properties of the deconfinement transition in finite temperature pure Lattice Gauge Theories (LGT) can be described by a suitable effective action for the order parameter, the Polyakov loop [1]. The construction of this effective action necessarily involves some approximations. It is of crucial importance to choose these approximations so as to obtain effective actions simple enough to be easily studied (either with exact solutions, or with some mean field like technique) and, at the same time, rich enough to keep track of the whole complexity of the original gauge theory. In the past years this problem was addressed with several different approaches (see [2] for references and discussion). However a common feature of all these approaches was that the effective actions were always constructed neglecting the spacelike part of the action. As a consequence it was impossible to reach a consistent continuum limit for the critical temperature.

The aim of this contribution is to show that it is possible to avoid such a drastic approximation. We shall discuss a general framework which allows one to construct improved effective actions which take into account perturbatively (order by order in the spacelike coupling β_s) the spacelike part of the original gauge action and are *exact to all orders in the timelike coupling*.

Our approach is valid for any gauge group G and for any choice of lattice regularization of the gauge action (Wilson, mixed, heat kernel actions...). Moreover it can be extended, in principle, to all orders in β_s.

We shall only outline here the general strategy and show, as an example, some results obtained by taking into account the first order contribution in β_s in the case of the $SU(2)$ gauge model in (3+1) dimensions. Much more details and a complete survey of our approach can be found in [2].

2. CONSTRUCTION OF THE EFFECTIVE ACTION

Since we treat in a different way the spacelike and timelike parts of the action, we are compelled to use two different couplings β_s and β_t. We shall denote with $\rho^2 \equiv \beta_t/\beta_s$ the asymmetry parameter, with N_t (N_s) the size of the lattice in the timelike (spacelike) direction and with d the number of spacelike dimensions.

The first step is to expand in characters both the timelike and the spacelike part of the action. The expansion of the spacelike part is truncated at the chosen order in β_s; the timelike part is kept exact to all orders.

For the contribution due to the trivial representation term in the spacelike expansion (the "ze-

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roth order" approximation in β_s) the integration over the spacelike degrees of freedom is straightforward. The resulting effective action is that of a d dimensional spin model (the spins being the Polyakov loops of the original model) with next neighbour interactions only.

The terms of higher order β_s give rise to non trivial interactions among several Polyakov loops. For instance at the order β_s^2 the interaction involves the four Polyakov loops around a spacelike plaquette. The explicit form of these interactions can be written in terms of rather complicated group integrals, which can be solved by means of suitable Schwinger Dyson (SD) equations. The use of these SD equations is crucial for our whole construction since for these integrals (in which the Polyakov loops are kept as free variables) the usual techniques, developed for ordinary strong coupling expansions, are useless. We shall discuss in detail, in the next section, a simple example.

2.1. Schwinger-Dyson equations

As an example, let us study the integral which appears in the discussion of the term due to the adjoint representation in the character expansion of the spacelike action for the SU(2) model:

$$\int DU U_{\alpha\beta} U^{\dagger}_{\gamma\delta} \chi_j (U P_{\vec{x}+i} U^{\dagger} P^{\dagger}_{\vec{x}}) \equiv \delta_{\alpha\delta}\delta_{\gamma\beta}\mathcal{C}^{(j)}_{\alpha\beta}. \quad (1)$$

First of all, it follows from (1) that the non vanishing integrals in the l.h.s. depend only on $|U_{\alpha\beta}|^2$, and hence that $\mathcal{C}^{(j)}_{11} = \mathcal{C}^{(j)}_{22}$ and $\mathcal{C}^{(j)}_{12} = \mathcal{C}^{(j)}_{21}$.

To compute the matrix elements $\mathcal{C}^{(j)}_{\alpha\beta}$, we note that the matrix $\mathcal{C}^{(j)}$ can be expressed in terms of the integral

$$\begin{aligned} K^{(j)}(\theta_{\vec{x}}, \theta_{\vec{x}+i}) &= \int DU \chi_j (U P_2 U^{\dagger} P_1^{\dagger}) \\ &= \frac{1}{d_j} \chi_j(P_2) \chi_j(P_1^{\dagger}) \end{aligned} \quad (2)$$

through a system of two linear Schwinger–Dyson-like equations. Indeed, considering the integral $\int DU U_{\alpha\beta} \; U^{\dagger}_{\beta\alpha} \; \chi_j (U P_{\vec{x}+i} U^{\dagger} P^{\dagger}_{\vec{x}})$ we easily find that

$$\mathcal{C}^{(j)}_{11} + \mathcal{C}^{(j)}_{12} = K^{(j)}. \quad (3)$$

To construct a second independent equation, let us consider the integral

$$\int DU \chi_{\frac{1}{2}} (U P_{\vec{x}+i} U^{\dagger} P^{\dagger}_{\vec{x}}) \chi_j (U P_{\vec{x}+i} U^{\dagger} P^{\dagger}_{\vec{x}}). \quad (4)$$

On one hand we can write the character $\chi_{\frac{1}{2}}$ explicitly as a trace and express the integral in terms of the $\mathcal{C}^{(j)}_{\alpha\beta}$ by using eq.(1). On the other hand, the integral (4) can be written in terms of $K^{(j)}$ functions by using the basic SU(2) Clebsch-Gordan relation: $\chi_{\frac{1}{2}}\chi_j = \chi_{j+\frac{1}{2}} + \chi_{j-\frac{1}{2}}$. The resulting equation is:

$$\begin{aligned} 2\cos(\theta_{\vec{x}+i} - \theta_{\vec{x}})\mathcal{C}^{(j)}_{11} \quad &+ \quad 2\cos(\theta_{\vec{x}+i} + \theta_{\vec{x}})\mathcal{C}^{(j)}_{12} = \\ K^{(j-\frac{1}{2})} \quad &+ \quad K^{(j+\frac{1}{2})} \end{aligned} \quad (5)$$

where $\{e^{i\theta_{\vec{x}}}, e^{-i\theta_{\vec{x}}}\}$ are the eigenvalues of the Polyakov loop $P_{\vec{x}}$. Eq.s (3) and (5) form a set of two linear equations in the unknowns $\mathcal{C}^{(j)}_{11}$ and $\mathcal{C}^{(j)}_{12}$ whose solution is:

$$\begin{aligned} \mathcal{C}^{(j)}_{11} &= \frac{K^{(j-\frac{1}{2})} - 2\cos(\theta_{\vec{x}+i} + \theta_{\vec{x}})K^{(j)} + K^{(j+\frac{1}{2})}}{4\sin\theta_{\vec{x}+i}\sin\theta_{\vec{x}}} \\ \mathcal{C}^{(j)}_{12} &= -\frac{K^{(j-\frac{1}{2})} - 2\cos(\theta_{\vec{x}+i} - \theta_{\vec{x}})K^{(j)} + K^{(j+\frac{1}{2})}}{4\sin\theta_{\vec{x}+i}\sin\theta_{\vec{x}}} \end{aligned}$$

This solves the problem.

3. DISCUSSION OF THE RESULTS

The effective action obtained in the previous section describes a d dimensional spin model with complicated interactions and cannot be solved exactly. However several features of the model can be figured out rather easily. In particular, the deconfinement temperature can be estimate by using a mean field approximation. The results can then be used in two ways.

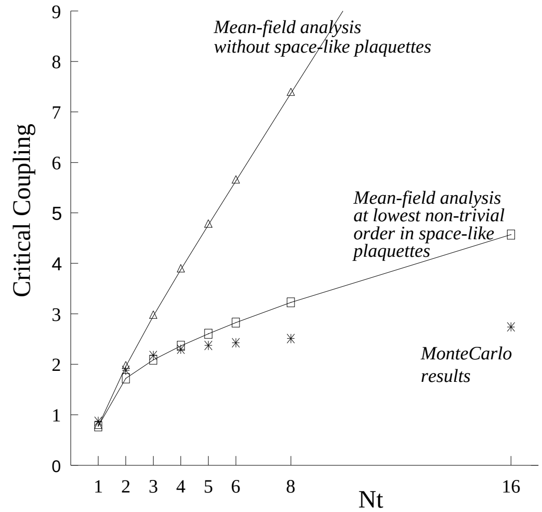
3.1. Asymmetric lattices

As ρ varies we have different, but equivalent, lattice regularization of the same model. This equivalence however implies, at the quantum level, in the (3+1) dimensional case a non trivial relation between the couplings. This problem was studied in the weak coupling limit by F. Karsch [3] who found, in the $\rho \to \infty$ limit, the following relation:

$$\beta_t = \rho\,(\beta + \alpha_t^0) + \alpha_t^1 \quad (6)$$

N_t	α_t^0	α_t^1
2	-0.184	0.414
3	-0.210	0.375
4	-0.221	0.373
5	-0.235	0.372
6	-0.249	0.389
8	-0.271	0.413
16	-0.327	0.508
	-0.27192	0.50

Tab. I *Values of of α_t^0 and α_t^1 as functions of N_t. The theoretical values are reported, for comparison, in the last row of the table.*



where β is the coupling of the equivalent symmetric regularization. α_t^0 and α_t^1 are group dependent constants whose value in the SU(2) case is: $\alpha_t^0 = -0.27192$ and $\alpha_t^1 = 1/2$;

The ρ dependence of our estimates for the critical temperature shows a remarkable agreement with the behaviour predicted by eq.(6) (see tab.I.). As N_t increases our estimates for α_t^0 and α_t^1 cluster around the theoretical values of [3]. This agreement is highly non trivial since α_t^0 and α_t^1 were obtained with a *weak coupling* calculation, while our effective action is the result of a *strong coupling* expansion. The reason of this success is very likely related to the fact that we have been able to sum to all orders in β_t the time-like contribution of the effective action.

3.2. Scaling behaviour

The second important test is the scaling behaviour of the deconfinement temperature as a function of N_t. The N_t dependence is predicted to be of logarithmic type in (3+1) dimensions and the Montecarlo data confirm this analysis. On the contrary all the effective actions obtained neglecting the spacelike plaquettes predict a linear scaling (see Fig.1). This was in past years one of the major drawbacks of the standard effective action approach to the deconfinement transition. The inclusion of the first non trivial corrections due to the space-like plaquettes greatly improves the scaling behaviour. The values obtained with our effective action for the critical couplings are plotted in Fig.1, where they are also compared with the Montecarlo results (extracted from [4]). Indeed, if one compares our mean field estimates with the ones of the Montecarlo simulations, it turns out that the discrepancy in critical couplings is within 10% in the range $2 \le N_t \le 5$. While the logarithmic scaling predicted by the renormalization group is still beyond the present scheme, being related to non perturbative effects in β_s, it is reasonable to expect that higher order approximations would lead to better and better numerical results at least for not too high values of N_t.

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